QUANTITATIVE FINANCE

School of Finance, SUFE

Exercise for Part III (Spring, 2020)

Dr. Luo, Dan

1. (St. Petersburg Paradox) The St. Petersburg paradox was introduced by Nicolaus Bernoulli in 1713. The standard version of the St. Petersburg paradox is derived from the St. Petersburg game, which is played as follows: A fair coin is flipped until it comes up heads the first time. At that point the player wins 2^n, where n is the number of times the coin was flipped. (For instance, if the coin lands heads on the first flip the player win 2, if it lands heads on the second flip then win 4, and if this happens on the third flip win 8, and so on.) What is the expected payoff of this game? For a player with the log utility function ($U(x) = \ln x$, where x is the random payoff of the game), what is her expected utility for playing this game? And what is her certainty equivalent value of this game?

2. (Relative Risk Premium) The Relative Risk Premium π_R is defined as the relative ratio the individual is willing to pay to avoid risk:

$$U(E[X](1 - \pi_R)) = E[U(X)] \tag{1}$$

   Derive the approximate expression of the relative risk premium using the Relative Risk Aversion Coefficient (Assuming $E[X] > 0$).

3. (Bid-Ask Spread) The reservation bid price, P_b, for a market maker can be defined through the following equation.

$$V(M - P_b, q + 1) = V(M, q), \tag{2}$$

   where $V(M, q)$ is the value function, that is, (maximized) expected utility, of holding cash M and q shares of a stock. Suppose $V(M, q) = E\left[-e^{-\gamma(M+qS)}\right]$, where $S \sim N(\mu, \sigma^2)$ denotes the stock price in next period. Try solving for P_b. The ask price, P_a, can be defined in a similar way. Try solving for P_a. Find the bid-ask spread and explain its determinants.

4. (Construction of a Representative Investor) Assume an individual investor chooses among n risky assets to maximize her expected utility as follows.

$$\max_{\omega_i} \quad \bar{R}_p - \frac{1}{\theta} Var(R_p), \tag{3}$$

where ω_i is the investor's portfolio weight in asset i such that $\sum_{i=1}^{n} \omega_i = 1$; The mean and variance of the investor's portfolio are $\bar{R}_p = \sum_{i=1}^{n} \omega_i \bar{R}_i$ and $Var(R_p) = \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_i \omega_j \sigma_{ij}$, respectively, with $\bar{R}_i$ being the expected return of asset i (with return R_i), and σ_{ij} being the covariance between the returns of assets i and j; θ is a positive constant and equals the investor's risk tolerance.

a. Write down the Lagrangian for this problem and derive the first-order conditions.

b. Rewrite the first-order condition to show that the expected return on asset i is a linear function of the covariance between the return of asset i and the return on the investor's optimal portfolio.

c. Assume that there are totally K investors in the market and investor k ($k = 1, 2, ..., K$) has (potentially different) initial wealth W_k and risk tolerance θ_k. For simplicity and without loss of generality, we assume $\sum_{k=1}^{K} W_k = 1$. Show that the equilibrium expected return on asset i is of a similar form to the first-order condition found in part (b), but depends on the wealth-weighted risk tolerance of all investors and the covariance of the return on asset i with that of the market portfolio (with return $R_M = \sum_{i=1}^{n} \left(\sum_{k=1}^{K} W_k \omega_i^k \right) R_i$). Hence the equilibrium asset pricing relations are the same as those from the hypothetical case wherein one representative investor with the wealth-weighted risk tolerance owns all the wealth. (Hint: Begin by multiplying the first order condition in (b) by the product of investor k's wealth and risk tolerance, and then aggregate over all investors.)